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The generalized harmonic oscillator and the infinite square well with a moving boundary

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Abstract. For the one-dimensional generalized harmonic oscillator we obtain in this paper its wavefunctions in closed form by means of two independent methods which are based respectively in (i) the algebraic properties of the dynamical symmetry of the system and (ii) the construction of an invariant operator. The total equivalence of these two formulations is shown and quantal properties are described in terms of a classical solution of the equations of motion. Two possible reductions for the system exist: the static harmonic oscillator and the free particle. In the latter case the quantum system becomes a Fermi oscillator or equivalently it can describe a free particle in a well with one moving boundary which in turn follows certain classical rules. The time-dependent boundary conditions in the well play the role of an effective interaction acting on the particle. The formalism is shown to be compatible with the gauge principle of minimal coupling and several different gauges are constructed and analysed.

1. Introduction

We consider in this work a one-dimensional generalized harmonic oscillator (GHO) described by the time-dependent Hamiltonian

$$H(t) = \beta_1(t)\frac{p^2}{2m} + \beta_2(t)\frac{\omega_0}{2}[x,\,p]_+ + \frac{1}{2}m\omega_0^2\beta_3(t)x^2 \tag{1.1}$$

where $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$ are three real functions of time. This system is a generalization of a harmonic oscillator with a mass or a frequency depending on time which have been extensively treated in [2–4, 8, 9]. We shall assume without loss of generality that the system is prepared at $t = 0$ in a state of the Hilbert space of the initial Hamiltonian. Two cases with quite different physical properties will be considered. (i) $\beta_1(0) = 1$, $\beta_2(0) = 0$, $\beta_3(0) = 1$ which initially reduces to a harmonic oscillator of frequency ω_0, and (ii) $\beta_1(0) = 1$, $\beta_2(0) = 0$, $\beta_3(0) = 0$, which initially reduces to a free particle.

Two independent methods to treat these problems shall be used. The first one [4] makes use of the properties of the dynamical symmetry of the system to build the temporal evolution operator and to obtain the instantaneous value of any meaningful physical property of the system and its temporal evolution. When this analysis is applied to an optical parametric amplifier [14, 19] which is a particular case of (1.1) with the same $SU(1,1)$ dynamical symmetry, exceedingly interesting physical properties of this system are found in a natural way: the existence of two photon states [19] with reduced quantum fluctuations and no classical statistics which may present antibunching [15]. These phenomena are determined by the dynamical symmetry and not by the specific form of the Hamiltonian and they are also present in other more general systems. For instance, GHO with several functional parameters [10]. The second method [4, 17] is based upon the construction of an invariant operator [9] whose eigenvectors are used to obtain the exact solution as well as the instantaneous physical state of the system. This method yields the wavefunctions [2] in a natural way in coordinate representation [8] and allows to analyse the nature of the non trivial quantal phases which are present in the system (Lewis and Berry phases) and its relative relationships among them [4, 8, 18]. Moreover, this method allows us to show that the evolution of these quantities with an intrinsically quantal nature is finally driven by a classical solution of the motion.

As a by-product of this research a new result will emerge, namely the relationship existing between a harmonic oscillator with time-dependent frequency and an infinite square well with a movable wall [7, 11]. The author believes that the connection between these two systems and its consequences have not been sufficiently explored and this is the main goal of this paper.

2. Wavefunctions and dynamical symmetry

The original system [4] can be identified as an Hermitian element of the $su(1, 1)$ Lie algebra. The characterization of this dynamical symmetry allows us (i) to find the instantaneous diagonalization the Hamiltonian and (ii) to construct the exact time-dependent evolution operator (TEO) in terms of a well-defined group element which exactly coincides with the action of a squeezing operator [14] generating the generalized $su(1, 1)$ coherent states of the group [12]:

$$S(\eta) = \mathrm{Exp}\{\eta K_+\}\,\mathrm{Exp}\{-2\log\cosh|\eta| K_0\}\,\mathrm{Exp}\{-\eta^* K_-\}. \tag{2.1}$$

The unitary operator $S(\eta)$ with characteristic parameters algebraically related to the parameters which define the physical system.

$$H(t) = 2\hbar\omega_0\sqrt{\beta_3\beta_1 - \beta_2^2}\{S(\eta)K_0 S^+(\eta)\} \tag{2.2}$$

$$\eta = \frac{\beta_1 - \beta_3 - 2\mathrm{i}\beta_2}{\beta_3 + \beta_1 + 2\sqrt{\beta_3\beta_1 - \beta_2^2}} \tag{2.3}$$

instantaneously diagonalize $H(t)$ if the functional parameters always satisfy the condition $\beta_3\beta_1 > \beta_2^2$. This is the only relevant condition [19] because it guarantees a unitary connection of (1.1) with a harmonic oscillator. However, the case with a zero root in (2.2) is also meaningful and it must correspond to a unitary reduction of the system to a free particle. This case has not been treated yet and we shall return to this point later on.

The TEO can also be obtained as

$$U(t) = S(\eta(t))\,\mathrm{Exp}\{\mathrm{i}h(t)K_0\} \tag{2.4}$$

where the complex function $\eta(t)$ is a solution of an ordinary Ricatti differential equation that yields, in turn, the real function $h(t)$ through a quadrature [4]:

$$\dot\eta = -\mathrm{i}\omega_0(f + f_0\eta + f^*\eta^2) \qquad \eta(0) = 0. \tag{2.5}$$

As soon as the functions $\eta(t)$ and $h(t)$ are known the physical properties of the system for a given initial state are determined. Not only the statistic of photons or fluctuations in any quadrature but even the wavefunctions in the system can be found using these quantities as an input. We shall then consider the time-dependent exact wavefunctions which evolve from several initial states of a static oscillator with frequency ω_0.

- *Number state.* The squeezing operator introduces a Bogolyubov transformation on the initial particles creating pseudoparticles which are created and annihilated by the operators

$$S(\eta)aS^+(\eta) = [S(\eta)a^+S^+(\eta)]^+ = \frac{a - \eta a^+}{\sqrt{1 - |\eta|^2}}. \tag{2.6}$$

The eigenstates of the new annihilation operator $S(\eta)aS^+(\eta)$ are the two photons coherent states [19] and making use of the coordinate representation of these operators we can obtain the exact wavefunctions of the GHO which may be finally expressed as

$$\Psi_n(x,t) = \left[\frac{\epsilon^2(t)}{\pi}\right]^{\frac{1}{4}} \mathrm{Exp}\left\{-\mathrm{i}\left(n + \frac{1}{2}\right)\int_0^t \omega(s)\,\mathrm{d}s\right\} \mathrm{Exp}\left\{\mathrm{i}\epsilon^2(t)\frac{\mathrm{Im}\,(\eta)}{(1+\eta)(1+\eta^*)}x^2\right\}$$
$$\times\, \mathrm{Exp}\left\{-\frac{\epsilon^2(t)}{2}x^2\right\}\frac{H_n[\epsilon(t)x]}{\sqrt{n!2^n}}. \tag{2.7}$$

- *Coherent state.* The previous wavefunctions and the generating function of the Hermite polynomials may be used to obtain in a closed form the wavefunction corresponding to an initial coherent state:

$$\Psi_\alpha(x,t) = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{q=0}^{\infty}\frac{\alpha^q\Psi_q(x,t)}{\sqrt{q!}} = \frac{\epsilon^{1/2}(t)}{\pi^{1/4}}\mathrm{Exp}\left\{\frac{1}{2}\left[\alpha^2(t) - |\alpha|^2 - \mathrm{i}\int_0^t \omega(s)\,\mathrm{d}s\right]\right\}$$
$$\times\,\mathrm{Exp}\left\{\mathrm{i}\frac{\epsilon^2(t)\mathrm{Im}\,(\eta)}{(1+\eta)(1+\eta^*)}x^2\right\}\mathrm{Exp}\left\{-\frac{\epsilon^2(t)}{2}\left[x - \frac{\sqrt{2}\alpha(t)}{\epsilon(t)}\right]^2\right\}. \tag{2.8}$$

Notice that in both cases, apart from the time-dependent phase factor, all these functions can be obtained starting with the correspondent wavefunction of a static oscillator and using the simple correspondence:

$$\epsilon \quad\longrightarrow\quad \epsilon(t) = \epsilon\sqrt{\frac{1 - |\eta(t)|^2}{[1+\eta(t)][1+\eta^*(t)]}} \tag{2.9}$$

$$\alpha \quad\longrightarrow\quad \alpha(t) = \alpha\exp\left\{-\mathrm{i}\int_0^t \omega(s)\,\mathrm{d}s\right\} \tag{2.10}$$

$$\omega \quad\longrightarrow\quad \omega(t) = \frac{\mathrm{d}\{\mathrm{Arg}[1+\eta(t)]\}}{\mathrm{d}t} - \frac{\dot{h}(t)}{2} \tag{2.11}$$

and those are reduced to these when η is zero. The time-dependent parameter $\epsilon(t)$ determines the squeezing properties of the system [4].

The evolution of an initial state when the time-dependent Hamiltonian is reduced initially to a free particle may be treated in a similar way. However, the factorization of equation (2.4) for $U(t)$ is not adequate in this case. The calculation of the evolution of the states with a well-defined momentum eigenvalue as $\langle x|U(t)|p\rangle$ is much easier if the second method mentioned above is used.

3. The invariant operator and the wavefunctions. The case $c \neq 0$

In 1969 Lewis and Riesenfeld [9] were able to show that given a physical system with a time-dependent Hamiltonian admitting a hermitic invariant operator $I(t)$, then, the eigenvalues λ of this invariant operator must be constants and the phases of its eigenstates $|\lambda, t >$ may be chosen in such a way that the linear superposition of states

$$|\Psi(t)\rangle = \sum_\lambda C_\lambda \mathrm{e}^{\mathrm{i}\alpha_\lambda(t)}|\lambda, t\rangle \tag{3.1}$$

constitutes the exact solution of the time-dependent Schrödinger equation for the Hamiltonian $H(t)$. $\alpha_\lambda(t)$ are the *Lewis phases* [9] and the constants C_λ are determined by the initial state of the system. Therefore, if the eigenvectors of this invariant operator and its correspondent wavefunctions $\Phi_\lambda(x,t)$ are known the exact wavefunction of the original system can be built as a superposition of such an eigenfuctions with its adequate phase in the form:

$$\Psi_\lambda(x,t) = \mathrm{e}^{\mathrm{i}\alpha_\lambda(t)}\Phi_\lambda(x,t). \tag{3.2}$$

The advantage of this method lies in the fact that one can transform the dynamical problem which deals with building an exact wavefunction for a given time-dependent Hamiltonian in another purely stationary problem which deals in turn with finding the eigenfunctions of the invariant operator. The aim is thus reduced to find the invariant operator for (1.1) together with its eigenvalues and eigenfunctions. The operator $I(t)$ (3.3)

$$I(t) = \frac{cte}{\beta_1}\left\{\beta_1^2\sigma^2p^2 - m\beta_1\Lambda\sigma^2[x,p]_+ + \left(\frac{c^2}{\sigma^2} + m^2\Lambda^2\sigma^2\right)x^2\right\} \tag{3.3}$$

where $\Lambda = \frac{\dot\sigma}{\sigma} + \frac{\dot\beta_1}{2\beta_1} - \omega_0\beta_2$, c is an arbitrary integration constant and $\sigma(t)$ a *real function* satisfying the Pinney ordinary differential equation [13]:

$$\frac{c^2}{m^2\sigma^3} = \ddot\sigma + \Omega^2(t)\sigma \tag{3.4}$$

is an invariant operator for $H(t)$ when the real time-dependent term $\Omega^2(t)$ [4] was the characteristic *frequency* of the correspondent *classical* system. This invariant has dimensions of energy if σ has a dimension of $M^{-1/2}$ and c is a typical frequency. The constant c with dimensions of frequency is in principle arbitrary but, according to [9], *all its values are equivalent and they describe the same physical system*. This assertion is completely true just if it is assumed that c is a non-vanishing constant. In fact, this is simply the only case which has been considered in the Lewis–Riesenfeld original work and all subsequent papers on this problem. But there is no reason why c cannot be zero [16]. In fact the two cases are possible and they correspond to two different physical systems.

It is interesting to notice that the existence and construction of the invariant operator is nearly related to the existence of a dynamical symmetry in the system. $I(t)$ has the same dynamical symmetry as the Hamiltonian (1.1) and can also be treated in the same way. In particular, it may be instantaneously diagonalized [5] in a proper basis of the Cartan subalgebra generator K_0 which represents an static harmonic oscillator. For any non-vanishing c, the invariant $I(t)$ can be obtained by acting with a unitary operator on a static harmonic oscillator with *arbitrary* constant frequency ω. Actually, the time-dependent unitary operator

$$W(t) = \mathrm{Exp}\left\{-\frac{\mathrm{i}}{4\hbar}\log\left\{\frac{m\omega\beta_1(t)\sigma^2(t)}{c}\right\}[x,p]_+\right\}\mathrm{Exp}\left\{\frac{\mathrm{i}}{2\hbar}\left\{\frac{m^2\omega\Lambda(t)\sigma^2(t)}{c}\right\}x^2\right\} \tag{3.5}$$

uniquely relates the Lewis–Riesenfeld invariant with a harmonic oscillator with a constant frequency ω for a fixed value of the dimensionles constant $\frac{\omega}{2c}$ in the form:

$$I(t) = W(t)\left\{\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\right\}W^+(t). \tag{3.6}$$

Therefore, the eigenstates of the invariant operator are obtained by the action of $W(t)$ on the static eigenvectors of the stationary harmonic oscillator in the form $|\lambda,t\rangle = |n,t\rangle = W(t)|n\rangle$. The states $|\lambda,t\rangle$ have a well-defined number of photons and may be labelled with the non-negative integers. The constant eigenvalues λ of $I(t)$ are given by $\lambda = \hbar\omega(n+\frac{1}{2})$ and its eigenfunctions are those of the harmonic oscillator.

The unitary operator $W(t)$ that instantaneously diagonalizes $I(t)$ is also an element of the $SU(1,1)$ Lie group and the methods in [6] can be applied to also characterize $W(t)$ in terms of a squeezing operator [14] in the form:

$$W = S(\eta_0)\,\mathrm{Exp}\{\mathrm{i}h_0K_0\} \tag{3.7}$$

$$\eta_0 = -\frac{c\omega_0 - m\omega\omega_0\beta_1\sigma^2 - \mathrm{i}m\omega\Lambda\sigma^2}{c\omega_0 + m\omega\omega_0\beta_1\sigma^2 - \mathrm{i}m\omega\Lambda\sigma^2} \tag{3.8}$$

$$h_0 = 2\arctan\frac{m\omega\Lambda\sigma^2}{\omega_0(c + m\omega\beta_1\sigma^2)}. \tag{3.9}$$

As soon as the invariant $I(t)$ has been built and the correspondent unitary transformation $W(t)$ has been constructed through the procedure just outlined, we turn our attention to the two free constants, namely the non-vanishing integration constant c and the static oscillator frequency ω which is also arbitrary. Let us assign to them certain specific values of interest. For the choice $\omega = \omega_0$ the static oscillator coincides with the initial oscillator. Moreover, a direct calculation shows that the time evolution of the parameter η_0 characterizing $W(t)$ satisfies:

$$\dot{\eta}_0 + \mathrm{i}\omega_0(f + f_0\eta_0 + f^*\eta_0^2) = \frac{2\mathrm{i}c^2\omega_0\beta_1(\omega^2 - \omega_0^2)}{(c\omega_0 + m\omega\omega_0\beta_1\sigma^2 - \mathrm{i}m\omega\Lambda\sigma^2)^2} \tag{3.10}$$

and by imposing for $\sigma(t)$ the initial conditions $\sigma(0) = \sqrt{\frac{c}{m\omega_0}}$ and $\dot{\sigma}(0) = -\frac{\dot{\beta}_1(0)}{2}\sqrt{\frac{c}{m\omega_0}}$, then $\eta_0(t)$ and $\eta(t)$ in the Riccati equation (2.5) coincide. Also, $W(t)$ verifies $W(0) = 1$, $I(0) = H(0)$ and initially the invariant and the Hamiltonian operators coincide. The constant eigenvalues of $I(t)$ can be identified as the energy spectrum of the initial Hamiltonian and the initial eigenstates of $I(0)$ in the superposition (3.1), determine a Hilbert space of states for the initial Hamiltonian which can be identified as the natural initial set of states in which the system has been prepared.

The operator $W(t)$ can be used for determining the eigenvectors, the eigenfunctions and the exact Lewis phases. With the spectrum of $I(t)$ and the form of $W(t)$, one can find the eigenfunctions of $I(t)$, by using $\Phi_n(x,t) = \langle x|W(t)|n\rangle$. In this case, however, it is easier to solve directly the differential equation relative to the eigenfunctions of (3.3)

$$\left\{-\hbar^2\beta_1\sigma^2\frac{\partial^2}{\partial x^2} + \mathrm{i}\hbar m\Lambda\sigma^2\left(2x\frac{\partial}{\partial x} + 1\right) + \frac{1}{\beta_1}\left(\frac{c^2}{\sigma^2} + m^2\Lambda^2\sigma^2\right)x^2\right\}\Phi_n(x,t) = 2\hbar c(n + \tfrac{1}{2})\Phi_n(x,t) \tag{3.11}$$

and then calculate the correspondent Lewis phase [4]. We finally obtain the set of orthonormalized eigenfunctions at any time over the real line:

$$\Psi_n(x,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{c}{\hbar\pi\beta_1\sigma^2}\right)^{\frac{1}{4}}\mathrm{Exp}\left\{-\frac{\mathrm{i}c}{m}\left(n + \frac{1}{2}\right)\int_0^t\frac{\mathrm{d}s}{\sigma^2(s)}\right\} \times \mathrm{Exp}\left\{\frac{\mathrm{i}m}{2\hbar\beta_1}\left(\Lambda + \frac{\mathrm{i}c}{m\sigma^2}\right)x^2\right\}H_n\left(\frac{\sqrt{c}x}{\sqrt{\hbar\beta_1\sigma^2}}\right) \tag{3.12}$$

with $H_n(z)$ a Hermite polinomial of order n in the variable $z = \frac{\sqrt{c}x}{\sqrt{\hbar\beta_1\sigma^2}}$ which depends on a real function $\sigma(t)$ which is a solution of Pinney's diferential equation (3.4). The wavefunction corresponding to a coherent initial state in this formalism now reads:

$$\Psi_\alpha(x,t) = \left(\frac{c}{\hbar\pi\beta_1\sigma^2}\right)^{\frac{1}{4}}\mathrm{Exp}\left\{\frac{1}{2}\left[\alpha^2(t) - |\alpha|^2 - \mathrm{i}\int_0^t\omega(s)\,\mathrm{d}s\right]\right\}$$

$$\times\,\mathrm{Exp}\left\{\frac{\mathrm{i}m\Lambda}{2\hbar\beta_1}x^2\right\}\mathrm{Exp}\left\{-\frac{c}{2\hbar\beta_1\sigma^2}\left[x-\sqrt{\frac{2\hbar\beta_1\sigma^2}{c}}\alpha(t)\right]^2\right\} \tag{3.13}$$

where $\alpha(t) = \alpha\exp\{-\mathrm{i}\int_0^t \omega(s)\,\mathrm{d}s\}$ and $\omega(t) = c\{m\sigma^2(t)\}^{-1}$.

The form of the wavefunctions (3.12)and (3.13) is especially interesting in regard to the nature of the function $\sigma(t)$. As is well known [1] the general solution to Pinney's differential equation can be built by means of two independent solutions, $\sigma_1(t)$ and $\sigma_2(t)$, satisfying the following homogeneous differential equation:

$$\ddot{\sigma} + \Omega^2(t)\sigma = 0 \tag{3.14}$$

which is actually *the classical equation of motion*. In fact, we only need one of these functions, (i.e. $\sigma_1(t)$ satisfying $\sigma_1(0) = 1$, $\dot{\sigma}_1(0) = 0$) because the other one can be easily obtained in a quadrature. The function we are looking for can be obtained by means of a nonlinear superposition law of the form

$$\sigma(t) = \sigma_1(t)\sqrt{\sigma_0^2 + 2\sigma_0\dot{\sigma}_0\rho(t) + \left(\dot{\sigma}_0^2 + \frac{c^2}{m^2\sigma_0^2}\right)\rho^2(t)} \tag{3.15}$$

with σ_0 and $\dot{\sigma}_0$ as the initial values and $\rho(t) = \int_0^t \frac{\mathrm{d}s}{\sigma_1^2(s)}$. In this way, the wavefunctions can be obtained just by finding *one classical solution of the equation of motion* $\sigma_1(t)$.

Obviously the two sets of wavefunctions (2.7), (2.8) and (3.12), (3.13) must be the same. The two formulations yield identical functions, each one emphasizing different properties of the system. One set depends upon the function $\eta(t)$ which carries on specifically the properties referred to as the quantum noise of the states. The other set depends on the real function $\sigma(t)$ which is attached to the classical trajectory of the system. The transformation among them is given by the following sets of equations:

$$\eta = -\frac{c - m\omega_0\beta_1\sigma^2 - \mathrm{i}m\Lambda\sigma^2}{c + m\omega_0\beta_1\sigma^2 - \mathrm{i}m\Lambda\sigma^2} \tag{3.16}$$

$$\sigma = \sqrt{\frac{c}{m\omega_0\beta_1}\frac{(1+\eta)(1+\eta^*)}{(1-|\eta|^2)}}. \tag{3.17}$$

Note that the quantity introduced above $\epsilon(t) = \sqrt{c\hbar^{-1}\beta_1^{-1}}\sigma^{-1}$ is a typical length of the system that yields information on the squeezing properties in terms of the classical motion. Moreover a close connection between the master operators $U(t)$ and $W(t)$ of each one of these two formulations can be shown to exist [4]:

$$W(t) = U(t)\,\mathrm{Exp}\left\{\frac{2\mathrm{i}c}{m}\int_0^t \frac{\mathrm{d}s}{\sigma^2(s)}K_0\right\}. \tag{3.18}$$

The time evolution operator differs from the operator that instantaneously diagonalizes the invariant $I(t)$ just in a term which acts on the eigenstates of the static oscillator adding the Lewis phases.

4. The invariant operator and the wavefunctions. The case $c = 0$

Let us now consider Pinney's equation (3.4) for the $c = 0$ case following a similar scheme as case $c \neq 0$. The transformation $W(t)$ (3.5) is singular. In this case one cannot choose a harmonic oscillator with the same frequency to the initial as the auxiliar oscillator in the system. However, if one chooses $\omega = 2c$ the singularity disappears and the correspondent transformation $W(t)$ is shown to be independent of c and the analysis can also be extended to the limit $c = 0$ case. Actually the operator

$$W(t) = \text{Exp}\left\{-\frac{\mathrm{i}}{4\hbar}\log(2m\beta_1\sigma^2)[x,\, p]_+\right\}\text{Exp}\left\{\mathrm{i}\frac{m^2\Lambda\sigma^2}{\hbar}x^2\right\} \tag{4.1}$$

connects in the usual unitary way (3.6) the Lewis–Riesenfeld invariant (3.3) with a $2c$ frequency static oscillator and a proportionality constant equal to one. This is true for any value of c. If we then set $\omega = 2c$ and consider the limit case $c = 0$ we conclude that it is possible to build a Lewis–Riesenfeld invariant for the system (1.1) in terms of a real solution of the classical equation of motion (3.14). This invariant is not unitarily related to a static harmonic oscillator but it is related to the *free particle* Hamiltonian:

$$I(t) = \sigma^2\left\{\sqrt{\beta}_1 p - m\frac{\Lambda}{\sqrt{\beta}_1}x\right\}^2 = W\frac{p^2}{2m}W^+ \tag{4.2}$$

which could indeed be recovered from the initial Hamiltonian with the following choice of initial conditions $\sigma(0) = \frac{1}{\sqrt{2m}}$ and $\dot\sigma(0) = -\frac{\dot\beta_1(0)}{2\sqrt{2m}}$. The eigenstates of this invariant are thus only obtained from momentum eigenstates. Therefore, in this case, the only restriction on the eigenvalues of $I(t)$ is that they must be positive and must be expressed in the form $\frac{\lambda^2}{2m}$ where λ is any real number. The invariant eigenfunctions are now readily calculated as

$$-\frac{\sigma}{\sqrt{\beta}_1}\left\{\mathrm{i}\hbar\beta_1\frac{\partial}{\partial x} + m\Lambda x\right\}\Phi_\lambda(x,t) = \frac{\lambda}{\sqrt{2m}}\Phi_\lambda(x,t) \tag{4.3}$$

$$\Phi_\lambda(x,t) = \Phi_\lambda^{(0)}(t)\,\text{Exp}\left\{\frac{\mathrm{i}}{\hbar}\left(\frac{m\Lambda}{2\beta_1}x^2 + \frac{\lambda}{\sqrt{2m\beta_1\sigma^2}}x\right)\right\}. \tag{4.4}$$

One has also to calculate the exact Lewis phases. Although one could in principle follow the same method which has been used in the case $c \neq 0$ [4] it seems simpler to directly solve the correspondent wave equation bearing in mind that we have only to identify in this function the term $\Phi_\lambda^{(0)}(t)$ whose argument corresponds precisely to this phase:

$$\mathrm{i}\hbar\frac{\dot\Phi_\lambda^{(0)}(t)}{\Phi_\lambda^{(0)}(t)} = \frac{\lambda^2}{4m^2\sigma^2} - \frac{\mathrm{i}\hbar}{2}\left\{\frac{\dot\sigma}{\sigma} + \frac{\dot\beta_1}{2\beta_1}\right\} \tag{4.5}$$

$$\Phi_\lambda^{(0)}(t) = \frac{1}{\sqrt{\sigma\sqrt{\beta}_1}}\,\text{Exp}\left\{-\frac{\mathrm{i}\lambda^2}{4m^2\hbar}\int_0^t\frac{\mathrm{d}s}{\sigma^2(s)}\right\}. \tag{4.6}$$

The new set of non-normalizable wavefunctions labelled by the real number λ now reads

$$\Psi_\lambda(x,t) = \frac{1}{\sqrt{\sigma\sqrt{2m\beta_1}}}\,\text{Exp}\left\{-\frac{\mathrm{i}\lambda^2}{4m^2\hbar}\int_0^t\frac{\mathrm{d}s}{\sigma^2(s)}\right\}\text{Exp}\left\{\frac{\mathrm{i}}{\hbar}\left\{\frac{m\Lambda}{2\beta_1}x^2 + \frac{\lambda}{\sqrt{2m\beta_1\sigma^2}}x\right\}\right\} \tag{4.7}$$

$\sigma(t)$ being a solution of the classical equation of motion. They also form a set of exact wavefunctions for the GHO and they represent the time evolution of plane waves as initial states. The discrete sum (3.1) must now be replaced by a continuous one and describes *the exact instantaneous eigenstate of an initially free particle which is now submitted to an interaction* $H(t)$. This state is determined by the solution of the classical motion represented by the function $\sigma(t)$.

Of course the close connection [4] among the two formulations is also maintained here. The operator $W(t)$ can be expressed in terms of a squeezing operator in a similar manner to (3.7) and η_0 and h_0 are the same (3.8), (3.9) with $\omega = 2c$. Note, however, that if $\sigma(t)$ is a solution of the homogeneous differential equation (3.14) $\eta_0(t)$ can no longer be identified as a solution of the Riccati equation (2.5). The equivalence between these two formulations must now be expressed by means of

$$U(t) = W(t)\,\mathrm{Exp}\left\{-\frac{\mathrm{i}}{4m^2\hbar}\int_0^t \frac{\mathrm{d}s}{\sigma^2(s)}p^2\right\} = S(\eta)\,\mathrm{Exp}\{\mathrm{i}hK_0\} \tag{4.8}$$

and the factorization formulae for these operators [6] allow us now to identify

$$\eta = -\frac{\omega_0 - 2m\omega_0\beta_1\sigma^2 - \mathrm{i}2m\Lambda\sigma^2 + \omega_0\sigma^2(\Lambda - \mathrm{i}\omega_0\beta_1)\int_0^t \frac{\mathrm{d}s}{\sigma^2(s)}}{\omega_0 + 2m\omega_0\beta_1\sigma^2 - \mathrm{i}2m\Lambda\sigma^2 + \omega_0\sigma^2(\Lambda + \mathrm{i}\omega_0\beta_1)\int_0^t \frac{\mathrm{d}s}{\sigma^2(s)}}. \tag{4.9}$$

Just as the case $c \neq 0$ [4], a direct calculation also shows in this case that the complex function $\eta(t)$ of (4.9) is a solution of the nonlinear differential equation (2.5) only when the real function $\sigma(t)$ is a solution of the linear differential equation (3.14).

5. The square well with a moving boundary

The facts described so far will acquire a clear physical meaning that shall be discussed in this section. Let us first take $\beta_1 = 1$ and $\beta_2 = 0$. Furthermore, we shall be considering that this particular case of the GHO is confined in an infinite potential well of size L. The usual conditions in the border of the well give rise to the quantization of the values of λ which can now only take the values

$$\sin\frac{\lambda L}{\sqrt{2m\hbar}\sigma(t)} = 0 \qquad \lambda = n\pi\hbar\frac{\sqrt{2m}\sigma(t)}{L}. \tag{5.1}$$

Since the eigenvalue λ must be a constant the results obtained so far will only be compatible if the width of the well L and the function $\sigma(t)$ were proportional to each other. A simple relationship meeting all requirements is $L = L_0\sqrt{2m}\sigma(t)$. We conclude then that the right wall moves with time and its relative motion is governed by the function $\sigma(t)$. The correspondent wavefunctions can be expressed as

$$\Psi_n(x,t) = \sqrt{\frac{2}{L(t)}}\,\mathrm{Exp}\left\{-\frac{\mathrm{i}n^2\pi^2\hbar}{2m}\int_0^t\frac{\mathrm{d}s}{L^2(s)}\right\}\mathrm{Exp}\left\{\frac{\mathrm{i}m\dot{L}(t)}{2\hbar L(t)}x^2\right\}\sin\left\{\frac{n\pi x}{L(t)}\right\}. \tag{5.2}$$

This set of wavefunctions (5.2) possesses some remarkable properties:

- They are orthogonal for all times and they are instantaneously normalized in the well in spite of the fact that the conditions at the boundary are obviously time dependent.
- Apart from a local phase factor which can be set to one in the static case these functions can be built just starting from the wavefunctions of the static case. Moreover, they are reduced to the static well eigenfunctions in the case L = constant.
- Although we are dealing with a time-dependent system, a known invariant exists whose eigenvalues are constants and coincide with the energy eigenvalues of the static system:

$$I(t) = \frac{L^2(t)}{2mL_0^2}\left(p - m\frac{\dot{L}(t)}{L(t)}x\right)^2. \tag{5.3}$$

  The functions (5.2) are precisely the eigenfunctions of this invariant operator with a calculable phase correction.

- This invariant is unitarily related to the *free particle* Hamiltonian in the form (4.2) and the wavefunctions can also be transformed by means of the already constructed operator $W(t)$:

$$W(t) = \mathrm{Exp}\left\{-\frac{\mathrm{i}}{4\hbar}\log(\frac{L^2(t)}{L_0^2})[x,\, p]_+\right\}\mathrm{Exp}\left\{\frac{\mathrm{i}m}{2\hbar}\frac{L(t)\dot{L}(t)}{L_0^2}x^2\right\}. \tag{5.4}$$

One could consider surprising the fact that although our system is a time-dependent GHO, the set of functions which has been built is identical to those of the Fermi oscillator [7, 11]. So far this system has usually been described by means of a Hamiltonian with a purely kinetic term (a free particle) confined in a well with a movable wall just by imposing this last property simply as a boundary condition. As we shall show below such a choice is not obvious. There exist several ways to generalize the static well to a time-dependent well with one movable boundary. One of these ways is based on the set of functions (5.2). According to this scheme, the time-dependent physical system whose wavefunctions are those considered in (5.2) and the invariant operator given by (5.3) through (4.2) and (5.4), is the most natural candidate to describe the above-mentioned time-dependent well. However, if one follows the formalism systematically one should conclude, according to the ideas developed in section 3, that the Hamiltonian of this time-dependent well should read

$$H(t) = \frac{p^2}{2m} - \frac{1}{2}m\frac{\ddot{L}}{L}x^2. \tag{5.5}$$

This Hamiltonian would be the only one which would consistently correspond to a harmonic oscillator with a time-dependent frequency which is determined by the form of the changes in the time of the well's width. $H(t)$ is obviously reduced to that of the free particle for a static well. The time-dependent oscillator potential has to be included as a consequence of the existence of time-dependent boundary conditions. The term is a sort of effective interaction of the time-dependent well due just to the boundary conditions.

These two important modifications which we have to introduce in the static system: the presence of a local phase term in the wavefunctions and the time-dependent repulsive oscillator term in the Hamiltonian (for $\ddot{L}(t)$ positive) arise just from the systematical application of the formalism. However, the way to proceed is not unique. Actually one can select the parameters in a different way, for instance

$$\beta_1(t) = 1 \qquad \omega_0\beta_2(t) = \frac{\dot{L}(t)}{L(t)} \tag{5.6}$$

and going again through the whole formalism we would find the alternative set of wavefunctions

$$\Psi_n'(x,t) = \sqrt{\frac{2}{L(t)}}\,\mathrm{Exp}\left\{-\frac{\mathrm{i}n^2\pi^2\hbar}{2m}\int_0^t\frac{\mathrm{d}s}{L^2(s)}\right\}\sin\left\{\frac{n\pi x}{Ł(t)}\right\} \tag{5.7}$$

in instead of the set given by (5.2). Notice the absence of the local phase factor. The functions (5.7) can easily be shown to be the exact wavefunctions of the Hamiltonian:

$$H'(t) = \frac{1}{2m}\left(p + m\frac{\dot{L}(t)}{L(t)}x\right)^2 - \frac{1}{2}m\frac{\dot{L}^2(t)}{L^2(t)}x^2. \tag{5.8}$$

Although (5.5) and (5.8) represent apparently different physical systems they are in fact related by means of a gauge transformation. Actually one easily obtains (5.8) from (5.5) by using the following rule:

$$p \quad\longrightarrow\quad p + \frac{\partial q(x,t)}{\partial x} \qquad V(x,t) \quad\longrightarrow\quad V(x,t) + \frac{\partial q(x,t)}{\partial t}$$

which is an obvious gauge transformation where $q(x,t) = \frac{1}{2}m\frac{\dot{L}(t)}{L(t)}x^2$ and the unitary generator is the operator $G(x)$ given by:

$$G = \mathrm{Exp}\left\{\frac{\mathrm{i}m}{2\hbar}\frac{\dot{L}(t)}{L(t)}\right\}x^2. \tag{5.9}$$

The generator $G(x)$ is in fact the local phase. In this way both descriptions are not only compatible but they simply represent a different choice of the gauge describing the same physical system.

6. Conclusions

In this paper we have discussed several dynamical features of the time-dependent GHO. Making use of the algebraic properties of the underlying dynamical symmetry it is possible to identify the exact TEO and build the wavefunctions for relevant initial states in the system starting on a complex function $\eta(t)$ which is a solution of a Riccati differential equation. They can be obtained starting on the correspondent functions of a static oscillator and later on by applying a correspondence law which has been rigorously established.

From a different but not unrelated point of view, the Lewis–Riesenfeld method allows us to make the same calculations by means of the construction of an invariant operator whose eigenfunctions yield the exact wavefunctions of the system in terms of a real function $\sigma(t)$ which is finally reduced to a solution of the classical equations of motion. Two possible cases arise depending on the assigned value to an arbitrary integration constant c. For the $c \neq 0$ case the invariant operator is unitarily equivalent to a static oscillator and we again recover the wavefunctions of the previous section. For the $c = 0$ case the former equivalence no longer holds and a unitary equivalence with the free particle case emerges. The exact evolution of this case is also obtained using the fact that the new static Hamiltonian $H(t)$ has well-defined momentum states. In both cases there exists a close relationship among the relevant functions $\eta(t)$ and $\sigma(t)$ and their corresponding TEO. This correspondence has been rigorously established. This equivalence also allows us to interelate several interesting and exactly calculable quantal properties in terms of classical trajectories.

A second and unexpected consequence is obtained when one considers $c = 0$ and the oscillator is confined in an infinite potential well with a width $L(t)$. The normalization conditions in the limits of the well determine a set of wavefunctions which can be understood as describing the instantaneous state of a free particle in a well with one movable boundary. This system can also be reinterpreted as a time-dependent harmonic oscillator whose time-dependent frequency is precisely determined by the form in which the length of the well changes. The *rate of change* acts on the free particle as an *effective interaction*. Although several characterizations of this effective interaction are possible they are all related by a gauge transformation. This gauge equivalence is also identified and the correspondent global gauge elements are explicitly constructed.

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